SOLUTION EXERCISE SHEET 17

Exercise 1. This exercise is again a direct application of the residue theorem. First, we observe that $1 + z + z^2 = \frac{1-z^3}{1-z}$ for $z \neq 1$. From that it directly follows that the two roots of that polynomial are the two non trivial third roots of unity, given by $z_1 = \exp(2\pi i/3), z_2 = \exp(4\pi i/3)$. Thus we can write $1 + z + z^2 = (z - z_1)(z - z_2)$. As we have a simple path around the circle centered at the origin and of radius 2 we get by the residue theorem that

$$\int_{\partial D(0,2)} \frac{1}{1+z+z^2} dz = 2\pi i (\mathrm{Res}(\frac{1}{1+z+z^2}, z_1) + \mathrm{Res}(\frac{1}{1+z+z^2}, z_2)).$$

Clearly we have that $\mathrm{Res}(\frac{1}{1+z+z^2}, z_1) = \frac{1}{z_1 - z_2}$ and $\mathrm{Res}(\frac{1}{1+z+z^2}, z_2) = \frac{1}{z_2 - z_1}$. We conclude that

$$\int_{\partial D(0,2)} \frac{1}{1+z+z^2} dz = 2\pi i (\frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_1}) = 0.$$

Exercise 2. We begin by observing that the integral is well defined and absolutely convergent. To compute this integral we split the map $\cos^2(z)$ into two parts and use the residue theorem. First, observe that we have the identity

$$\cos^2(z) = \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 = \frac{e^{2iz} + 2 + e^{-2iz}}{4} = \frac{e^{2iz} + 2}{4} + \frac{e^{-2iz}}{4}.$$

The key observation is that e^{iz} is bounded on the upper half complex plane, i.e. on $\{\mathrm{Im}(z) \geq 0\}$ and e^{-iz} on the lower half complex plane. Now, consider the following path

$$\gamma_R = \begin{cases} R\exp(t\pi i), & t \in [0,1], \\ 2Rt - 3R, & t \in [1,2], \end{cases}$$

where $R > 1$. Then, using the residue theorem and observing that $\frac{e^{2iz}+2}{4(1+z^2)}$ has two simple singularities at $i, -i$ we get that

$$\int_{\gamma_R} \frac{e^{2iz} + 2}{4(1+z^2)} dz = 2\pi i \,\mathrm{Res}\left(\frac{e^{2iz} + 2}{4(1+z^2)}, i\right) = 2\pi i \frac{e^{-2} + 2}{4 \cdot 2i} = \pi \frac{e^{-2} + 2}{4}.$$

This holds for every $R > 1$. Furthermore, we can also decompose the integral as

$$\int_{\gamma_R} \frac{e^{2iz} + 2}{4(1+z^2)} dz = \int_{\gamma_{R,C}} \frac{e^{2iz} + 2}{4(1+z^2)} dz + \int_{-R}^{R} \frac{e^{2ix} + 2}{4(1+x^2)} dx,$$

where $\gamma_{R,C}$ is the simple parametrization for the upper half circle of radius $R > 1$ centered at the origin. By absolute convergence we clearly have that $\int_{-R}^{R} \frac{e^{2ix}+2}{4(1+x^2)} dx \xrightarrow{R \to \infty}$

$\int_{-\infty}^{\infty} \frac{e^{2ix}+2}{4(1+x^2)} dx$. Furthermore, we have that

$$\left| \int_{\gamma_{R,C}} \frac{e^{2iz}+2}{4(1+z^2)} dz \right| \leq \text{length}(\gamma_{R,C}) \sup_{z \in \gamma_{R,C}} \left| \frac{e^{2iz}+2}{4(1+z^2)} \right| \leq \pi R \frac{3}{4(R^2-1)} \xrightarrow{R\to\infty} 0.$$

We infer that

$$\int_{-\infty}^{\infty} \frac{e^{2ix}+2}{4(1+x^2)} dx = \pi \frac{e^{-2}+2}{4}.$$

Next, we need to compute $\int_{-\infty}^{\infty} \frac{e^{-2ix}}{4(x^2+1)} dx$. As $z \mapsto e^{-2iz}$ is bounded on the lower half complex plane, we follow the same idea then above only that this time we consider the path on the lower half circle of radius $R > 1$ and centered at the origin. We get that

$$-\int_{-\infty}^{\infty} \frac{e^{-2ix}}{4(x^2+1)} dx = 2\pi i \operatorname{Res}\left(\frac{e^{-2iz}}{4(z^2+1)}, -i \right) = 2\pi i \frac{e^{-2}}{4\cdot(-2i)} = \pi \frac{-e^{-2}}{4}.$$

In summary we have that

$$\int_{-\infty}^{\infty} \frac{\cos^2(x)}{(x^2+1)} dx = \pi \left(\frac{e^{-2}+2}{4} + \frac{e^{-2}}{4} \right) = \pi \frac{1+e^{-2}}{2} = \frac{\pi}{e}\cosh(1).$$

Exercise 3. We directly show the general case. First, observe that the integral is well defined since

$$\begin{aligned}(ax+bi)^2 &= (\operatorname{Re}(a)x + i(b+\operatorname{Im}(a)x))^2 \\ &= \operatorname{Re}(a)^2x^2 - \operatorname{Im}(a)^2x^2 - b^2 - 2b\operatorname{Im}(a)x + 2i\operatorname{Re}(a)(b+\operatorname{Im}(a)x),\end{aligned}$$

giving that

$$\operatorname{Re}((ax+bi)^2) = (\operatorname{Re}(a)^2 - \operatorname{Im}(a)^2)x^2 - 2b\operatorname{Im}(a)x - b^2, \quad \text{and}$$
$$|\exp(-(ax+bi)^2)| = \exp(-(\operatorname{Re}(a)^2 - \operatorname{Im}(a)^2)x^2 + 2b\operatorname{Im}(a)x + b^2).$$

As $t \in [0, \frac{\pi}{4})$, we have that $\operatorname{Re}(a) > \operatorname{Im}(a)$, which shows that the integral is well defined.

Now, to compute it, we first 'remove' b from the integral. More precisely, we show that

$$\int_{-\infty}^{\infty} e^{-(ax+bi)^2} dx = \int_{-\infty}^{\infty} e^{-a^2x^2} dx.$$

In order to achieve that we rewrite the integrant in a more connivent way, i.e. $e^{-(az+bi)^2 = e^{-a^2(z+c)}}$ with $c := \frac{bi}{a}$, then we define the following path,

$$\gamma_R = \begin{cases} -R+2Rt, & t\in[0,1], \\ R+c(t-1), & t\in[1,2], \\ R+c-2R(t-2), & t\in[2,3], \\ -R+c-c(t-3), & t\in[3,4], \end{cases}$$

where $R > 0$. By Cauchy's theorem we know that as $z \mapsto \exp(-a^2z^2)$ is holomorphic on $\mathbb{C}$, we have that $\int_{\gamma_R} \exp(-a^2z^2)dz = 0$. But also by decomposition of the integral, we have that

$$\int_{\gamma_R} \exp(-a^2z^2)dz = \int_{-R}^{R} \exp(-a^2x^2)dx + \int_{\gamma_{R,2}} \exp(-a^2z^2)dz$$
$$- \int_{-R}^{R} \exp(-a^2(z+c)^2)dx + \int_{\gamma_{R,4}} \exp(-a^2z^2)dz,$$

where $\gamma_{R,2}(t) = R + ct$ and $\gamma_{R,4}(t) = -R + c - ct$. We already know, that

$$\int_{-R}^{R} \exp(-a^2x^2)dx \xrightarrow{R\to\infty} \int_{-\infty}^{\infty} \exp(-a^2x^2)dx,$$
$$\int_{-R}^{R} \exp(-a^2(z+c)^2)dx \xrightarrow{R\to\infty} \int_{-\infty}^{\infty} \exp(-a^2(z+c)^2)dx.$$

Thus to conclude that we can 'remove' b, we only need to show that

$$\int_{\gamma_{R,2}} \exp(-a^2z^2)dz + \int_{\gamma_{R,4}} \exp(-a^2z^2)dz \xrightarrow{R\to\infty} 0.$$

But this is not difficult to see as

$$\left|\int_{\gamma_{R,2}} \exp(-a^2z^2)dz\right| + \left|\int_{\gamma_{R,4}} \exp(-a^2z^2)dz\right|$$
$$\leq \text{length}(\gamma_{R,2}) \sup_{z\in\gamma_{R,2}} |\exp(-a^2z^2)| + \text{length}(\gamma_{R,4}) \sup_{z\in\gamma_{R,4}} |\exp(-a^2z^2)|$$
$$\leq 2|c| \exp(-\operatorname{Re}(a^2)R^2 + 4|a||c|R + 2|c|^2|a|^2) \xrightarrow{R\to\infty} 0,$$

as for $t \in [0,1]$ we have

$$a^2z^2 = a^2(\pm R + ct)^2 = (\operatorname{Re}(a^2) + i\operatorname{Im}(a^2))(R^2 \pm 2ctR + c^2t^2)$$
$$\implies \operatorname{Re}(a^2z^2) = \operatorname{Re}(a^2)(R^2 \pm 2\operatorname{Re}(c)tR + \operatorname{Re}(c^2)t^2) - \operatorname{Im}(a^2)(\pm 2tR\operatorname{Im}(c) + t^2\operatorname{Im}(c^2))$$
$$\implies \operatorname{Re}(a^2z^2) \geq \operatorname{Re}(a^2)R^2 - 4|a||c|R - 2|c|^2|a|^2.$$

We conclude that we can remove b. Next, we compute directly the integral expression via change of variables. We have that

$$\left(\int_{-\infty}^{\infty} e^{-a^2x^2}dx\right)^2 = \int_{\mathbb{R}^2} \exp(-a^2(x^2+y^2))d(x,y)$$
$$= \int_0^{\infty}\int_{-\pi}^{\pi} r\exp(-a^2r^2)d\theta dr = \pi\int_0^{\infty} 2r\exp(-a^2r^2)dr$$
$$= \frac{\pi}{-a^2}\exp(-a^2r^2)|_{r=0}^{\infty} = \frac{\pi}{a^2}.$$

Finally, we conclude that $\int_{-\infty}^{\infty} e^{-a^2x^2}dx = \frac{\sqrt{\pi}}{a}$ if $a = re^{it}$ with $r > 0$ and $t \in [0, \frac{\pi}{4})$ and $\int_{-\infty}^{\infty} e^{-a^2x^2}dx = \frac{\sqrt{\pi}}{|a|}$ if $a \in \mathbb{R}_-^*$. This last part is clear if $a \in \mathbb{R}^*$. If $a = r\exp(it)$ with

$r > 0$ and $t \in [0, \frac{\pi}{4})$, we show that $\alpha \mapsto \int_{-\infty}^{\infty} e^{-\alpha x^2} dx$ is well defined and continuous on $\mathrm{Re}(\alpha) > 0$.

Exercise 4. This is a direct consequence of the product formula for the sinus. We know by product formula of the sinus, i.e. theorem 5.28 in the lecture notes, that

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right),$$

where the product converges locally uniformly on $\mathbb{C}$. Making the choice $z = \frac{1}{2}$ we get that

$$1 = \frac{\pi}{2} \prod_{n=1}^{\infty} \left(1 - \frac{1}{4n^2}\right) \iff \frac{\pi}{2} = \frac{1}{\prod_{n=1}^{\infty} \left(1 - \frac{1}{4n^2}\right)} \overset{(*)}{=} \prod_{n=1}^{\infty} \frac{1}{1 - \frac{1}{4n^2}} = \prod_{n=1}^{\infty} \frac{4n^2}{4n^2 - 1}.$$

How do you justify $(*)$?

Exercise 5. The goal of this exercise is to prove the product formula for the cosinus, i.e. that we have

$$\cos(\pi z) = \prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{(2n-1)^2}\right),$$

where the product converges locally uniformly on $\mathbb{C}$.

In order to do that we first observe that $\prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{(2n-1)^2}\right)$ is a well defined holomorphic function. Indeed, by exercise 3 of sheet 15, we have that the functions of functions $\left(\prod_{n=1}^{N} \left(1 - \frac{4z^2}{(2n-1)^2}\right)\right)_N$ converges locally uniformly if the sequence $\left(\frac{4z^2}{(2n-1)^2}\right)_n$ converges locally normally. As this is clearly the case, we know that the infinite product $\prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{(2n-1)^2}\right)$ is a well defined holomorphic map.

Then for $z \in \mathbb{C} \setminus \mathbb{Z}$, we observe that $\cos(\pi z) = \frac{\sin(2\pi z)}{2 \sin(\pi z)}$, giving that

$$\cos(\pi z) = \frac{\prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{n^2}\right)}{\prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)} = \prod_{n=1}^{\infty} \frac{1 - \frac{4z^2}{n^2}}{1 - \frac{z^2}{n^2}}.$$

Now, observe that

$$\prod_{n=1}^{\infty} \frac{1 - \frac{4z^2}{n^2}}{1 - \frac{z^2}{n^2}} = \lim_{N \to \infty} \prod_{n=1}^{2N} \frac{1 - \frac{z^2}{(n/2)^2}}{1 - \frac{z^2}{n^2}} = \lim_{N \to \infty} \frac{\prod_{n=1}^{N} 1 - \frac{4z^2}{(2n-1)^2}}{\prod_{N+1}^{2N} 1 - \frac{z^2}{n^2}} = \prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{(2n-1)^2}\right),$$

where we used that $\prod_{N+1}^{2N} (1 - \frac{z^2}{n^2}) \xrightarrow{N \to \infty} 1$ (why?).
We conclude that $\cos(\pi z) = \prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{(2n-1)^2}\right)$ for every $z \in \mathbb{C} \setminus \mathbb{Z}$. Then as both maps are holomorphic we infer that the equality holds for every $z \in \mathbb{C}$.